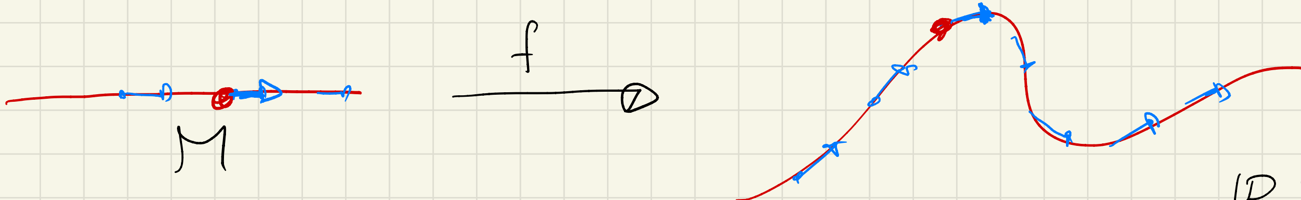
Lezione 7

Lemma: $f: M \hookrightarrow N$ embedding

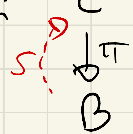
$X \in \mathfrak{X}(M) \rightsquigarrow Y(f(p)) = df_p(X(p))$

è un campo vettoriale su $f(M)$



$\mathbb{R}^2 = N$

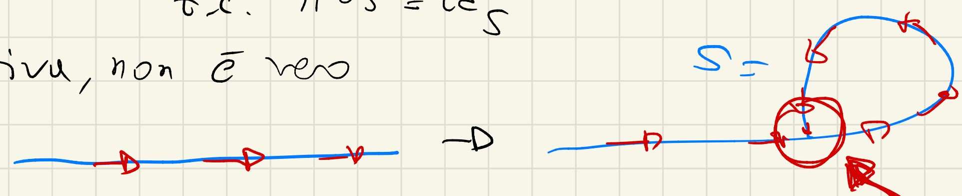
Sezione di E definita su $S \subseteq B$



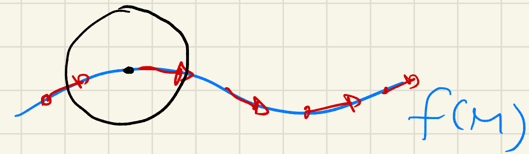
è $s: S \to E$ LISCIA

t.c. $\pi \circ s = id_S$

Oss: Se f è imm. iniettiva, non è vero



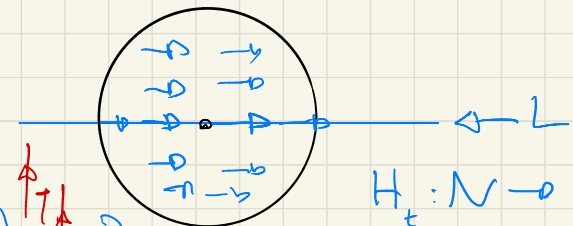
dim: $f(M) \subseteq N$ sottovarietà



Teo: $f, g : \underset{\text{CPT}}{M} \hookrightarrow N$ isotopi

$\Downarrow$

amb. isotopi



$H_t : N \to N$

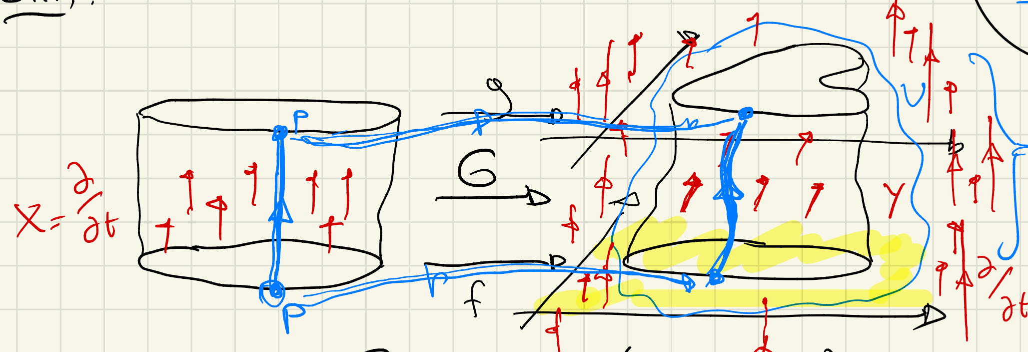
$p \mapsto H(p,t)$

DIFFEO

$H_0 = id$

$\boxed{H_1 \circ f = g}$?

dim:



$F_t : M \to N \qquad G(x,t) = (F_t(x), t) \qquad G : M \times \mathbb{R} \to N \times \mathbb{R}$

1) G embedding

2) $Y = dG(X)$ LO CONSIDERO SOLO SU $B = G(M \times [0,1])$ cpt

a)

b) Estendo Y in modo che sia nullo su $U \supseteq B$ $\quad \bar{U}$ cpt

c) Cambio in modo che la comp. verticale di Y sia $\frac{\partial}{\partial t}$

3) Y completo. Dentro $\bar{U}$ cpt $\Rightarrow \exists\, \varepsilon > 0$ funziona anche fuori U.

$\Phi_t : N \times \mathbb{R} \to N \times \mathbb{R}$ $\qquad \Phi_t(p, 0) = (H(p,t), t)$

è l'isotopia ambiente

Cor: M connessa è OMOGENEA cioè $\forall p, q \in M \ \exists\, \varphi \in \mathrm{Diffeo}(M)$ t.c. $\varphi(p) = q$

$\varphi = F_1$

dim: $\exists$ sempre $\alpha : [0,1] \to M$ t.c. $\alpha(0) = p$, $\alpha(1) = q$ $\quad$ (ex)

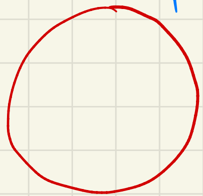
LISCIA

α è una isotopia fra $f, g : \{pt\} \to M$ $\quad f(pt) = p$, $g(pt) = q$

$\Rightarrow \exists$ isotopia ambiente che estende α
$F_t : M \to M$ $\quad F_0 = id$ $\quad F_1 \circ f = g$

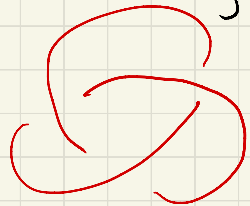
$\Rightarrow F_1 \circ f = g \Rightarrow F_1(p) = q$

f



$\mathbb{Z}$

g



$\neq \mathbb{Z}$

$g, f : S^1 \hookrightarrow \mathbb{R}^3$ non sono isotopi

se f e g isotopi $\Rightarrow$ $\mathbb{R}^3 \setminus f(S^1)$
$\cong \mathbb{R}^3 \setminus g(S^1)$

<u>Cor</u>: $f, g : \underset{\text{cpt}}{M} \hookrightarrow N$ emb. isotopi $\Rightarrow$ $N \setminus f(M) \cong N \setminus g(M)$
diffeomorfi

Infatti $\exists\ \varphi = F_1 : N \to N$ t.c. $g = \varphi \circ f$

$\varphi : f(M) \to g(M)$

$\varphi : N \setminus f(M) \to N \setminus g(M)$

PARENTESI DI LIE

$X \in \mathfrak{X}(M) = \Gamma TM$ $\quad f \in \mathcal{C}^\infty(M) = \mathcal{C}^\infty(M, \mathbb{R})$

$fX \in \mathfrak{X}(M)$ $\quad (fX)(p) = f(p)\, X(p)$

$Xf \in \mathcal{C}^\infty(M) \qquad (Xf)(p) = X(p)(f)$

Def: $X, Y \in \mathfrak{X}(M) \qquad [X,Y] \in \mathfrak{X}(M) \qquad [\ ,\ ]$ PARENTESI DI LIE

è l'unico campo per cui $\forall f \in \mathcal{C}^\infty(U)$ $U \subseteq M$ qualsiasi

$[X,Y](f) = XYf - YXf$ ☆ $(X,Y)f = XYf$ NON FUNZIONA

Prop: $[X,Y]$ è ben definito.

dim: "Devo mostrare che $[X,Y]$ è una derivazione in ogni punto."

$T_pM \ni [X,Y](p)(f) = (XYf - YXf)(p)$ ☆

Devo mostrare che $[X,Y](p)$ è derivazione

cioè $[X,Y](fg) = [X,Y](f) \cdot g + [X,Y](g) \cdot f$ TESI.

$$\begin{gathered}[X,Y](fg)\\ \| \\ (XY - YX)(fg)\end{gathered}$$

$$\begin{cases} \underbrace{X(\underbrace{Y(\underbrace{fg}))}} = \underbrace{X(\underbrace{\underbrace{Yf}\cdot g})} + X(f\cdot(Yg)) = \\ \qquad = X(\underbrace{Y f})\cdot g + \cancel{(Yf)\cdot(Xg)} + \cancel{(Xf)\cdot(Yg)} + f\cdot(X(Yg)) \\ YX(fg) = YXf\cdot g + \cancel{(Xf)(Yg)} + \cancel{(Yf)(Xg)} + f(YXg) \end{cases}$$

$$[X,Y](fg) = (XY-YX)(f)\cdot g + f\cdot(XY-YX)(g)$$

$$= [X,Y]f\cdot g + [X,Y]g\cdot f$$

$\Rightarrow [X,Y]$ è derivazione!

Def: Un' ALGEBRA DI LIE è sp. vett. A su $\mathbb{R}$ con una forma bil.

$[\ ,\ ] : A\times A \to A$ 1) antisimmetrica $[X,Y] = -[Y,X]$

2) identità di Jacobi

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

Ex: $\mathcal{X}(M)$ con $[\,,]$ formano algebra di Lie

Ex: In coordinate: X, Y campi in $\mathbb{R}^n$ $X = X^i \frac{\partial}{\partial x^i}$

$$\boxed{[X,Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}}$$

$[X,Y]f = (XY - YX)(f)$
$XYf - YXf$

$$[X,Y] = X^j \frac{\partial Y}{\partial x^j} - Y^j \frac{\partial X}{\partial x^j}$$

Cor: Se $X = \frac{\partial}{\partial x_i}$ costante $X^i = 1$ $X^j = 0$ $j \neq i$

$[X,Y] = \frac{\partial Y}{\partial x_i}$ Se anche $Y = \frac{\partial}{\partial x_i}$ allora $[X,Y] = 0$

Ex: $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$ campo

$X, Y \in \mathfrak{X}(M)$

$f, g \in \mathcal{C}^\infty(M)$

Ex: $A, B \in M(n)$ su $\mathbb{R}^n$ $\quad X(x) = A \cdot x \qquad Y(x) = B \cdot x$

$[X, Y](x) = (BA - AB)\, x$

Ricordiamo $[A, B] := AB - BA \qquad [a, b] = a b a^{-1} b^{-1}$

Def: $X, Y \in \mathfrak{X}(M)$ COMMUTANO se $[X, Y] = 0$

FLUSSI

Prop: Siano $X, Y \in \mathfrak{X}(M)$ e F, G i loro flussi (X, Y)

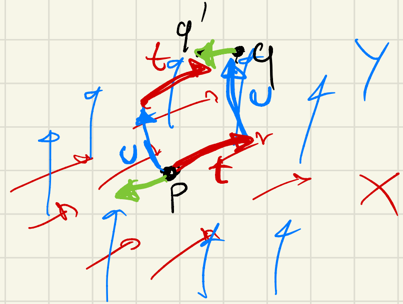
$F(p, t) = \gamma_p^X(t)$ ove definita $t \in I_p$

$F_t(p) = F(p, t)$

$G_u(p) = G(p, u)$

In carte OGNI CARTA

$$G_u \circ F_t(p) - F_t \circ G_u(p) = tu\,[X, Y](p) + o(t^2 + u^2)$$



$t,u \geq 0$

I flussi non commutano in generale

Prop: $X, Y \in \mathfrak{X}(M)$ commutano $\Longleftrightarrow$ F e G commutano

cioè $\underbrace{G_u(F_t(p))}_{} = \underbrace{F_t(G_u(p))}_{}$

per i valori p, t, u dove ha senso

dim:

$\boxed{\Longleftarrow}$ usa il lemma prec.

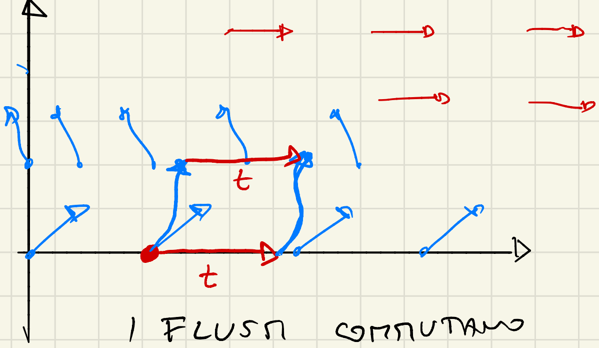
$\boxed{\Longrightarrow}$ $[X,Y](p) = 0 \overset{?}{\Longrightarrow}$ F e G commutano in p

a) Se $X(p) = 0$ e $Y(p) = 0$ $\Rightarrow F_t(p) = p$ $\Rightarrow G_u(p) = p$ OK

b) Se $X(p) \neq 0$ $\xrightarrow{\text{RADDRIZZ.}}$ In una carta $p = 0$, $X = \frac{\partial}{\partial x_1}$

$$X = \frac{\partial}{\partial x_1} \qquad Y = Y^i \frac{\partial}{\partial x_i}$$

$$\frac{\partial Y}{\partial x_1} = [X, Y] = 0$$



$\square$

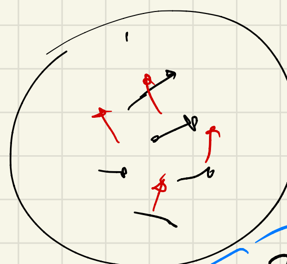
Prop: (RADDRIZZAMENTO SIMULTANEO)

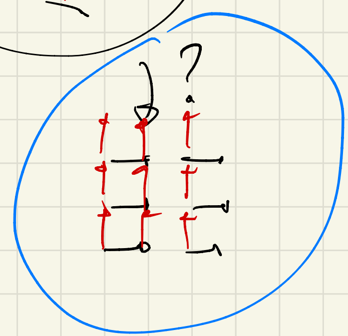
$X_1, \dots, X_k \in \mathfrak{X}(M)$

$X_1(p), \dots, X_k(p)$ indipendenti

$\exists$ una carta che li raddrizza vicino p $\Leftrightarrow$ $[X_i, X_j] = 0$ $\forall i, j$

(cioè che trasforma X_i in $\frac{\partial}{\partial x_i}$)





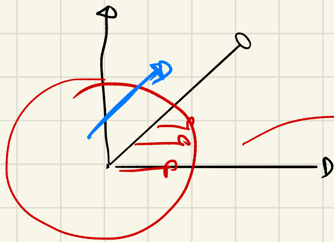
$\Rightarrow$ ovvio

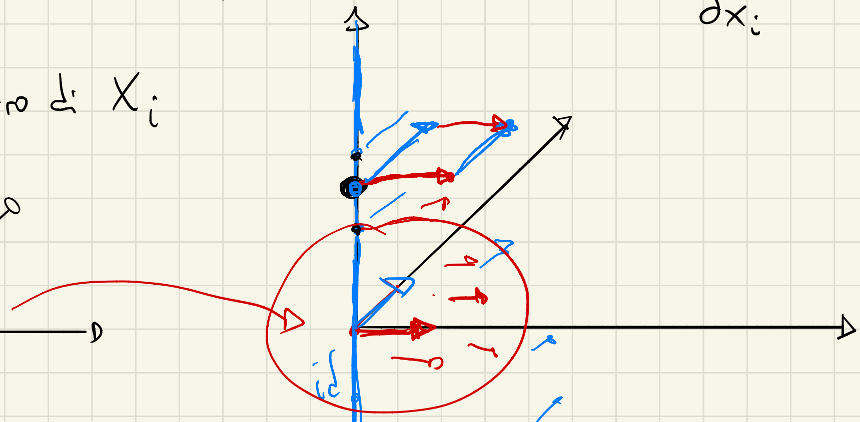
$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

Possiamo supporre $p = 0 \in \mathbb{R}^n$ $\quad X_i(0) = \frac{\partial}{\partial x_i}$

$\boxed{\Leftarrow}$

F^i_t flusso di X_i





$k=2$

$n=3$

$$\psi(x_1, \overset{x_k}{\dots}, x_n) = F^k_{x_k} \circ \dots \circ F^1_{x_1}(0, \dots, 0, x_{k+1}, \dots, x_n)$$

ψ è definita in un $U(0)$

$d\psi_0 = id$ (come dimo prec.)

ψ diffeo loc $\qquad \psi$ manda $x + te_k \to$ linea integrale di X_k $\square$